Automatic Choice of Global Shape Functions in Structural Analysis

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Linear analysis of highly complex structures is now feasible with standard finite-element programs. Nonlinear static or dynamic analysis of the same structures, however, often makes impossible demands on computer resources. A method is presented in which the nonlinear portions of the analysis are carried out in a reduced equation system, in which the generalized degrees of freedom are the coefficients in a summation of global shape functions. These functions are selected from the space of trial functions defined by a finite-element discretization. Important features of the method are: 1) rigorous treatment of the geometric nonlinear effects (complete coupling of global shape functions); 2) automatic problem-dependent choice of global shape functions; and 3) continuous monitoring of solution accuracy. The procedures are demonstrated by application in analysis of beams and arches.

Introduction

DESIGN of complex structures, in order to avoid stability failure, can be achieved through reliance on good analysis, good luck, and conservative design.

Before the high-speed computer was available and suitable software developed, it was hardly possible to perform a nonlinear, static or dynamic, analysis of anything but the very simplest structural components. Since luck is notoriously fickle, a conservative design was generally necessary, often with substantial penalties in weight and cost.

Today, much better computational resources are available, but this does not necessarily mean that a good analysis is always performed. Complex structures behave in complex ways; therefore, a good nonlinear analysis requires not only a good computer code but also participation of a well-trained analyst. Use of a sophisticated computer program without reasonable understanding of its theoretical background and the solution procedures involved may lead to an unjustified sense of security. The analyst may abandon the conservative approach because he has absolute confidence in his elaborate computer code, and a good deal of luck may be required if a catastrophe is to be avoided.

The main factors that now prevent the performance of a good nonlinear structural analysis appear to be: 1) lack of understanding of the basic concepts on the part of the analyst; 2) inadequate information about available software; 3) limitations in budget for computer runs; and 4) limitations in available software.

The problem of user education has recently been the subject of much discussion. The question of user certification has, at times, been raised. The problem remains unsolved, but presumably the number of available and competent analysts will increase with time. In the meantime, the problem can be mitigated by program developers through additional efforts to make program use easier.[1] Software retrieval systems can be developed and kept up-to-date.[2]

User education may improve the situation, but the development of more efficient software is equally important. More often than not only a bifurcation buckling analysis is relied on to determine the limit of structural stability. The analyst may be aware of the limitations of this approach,[3] but a nonlinear analysis is beyond his means. A nonlinear transient analysis over a time-interval long enough to reveal possible resonance or parametric excitation is prohibitively expensive if the number of degrees of freedom is more than a few hundred. The question of how many dollars could justifiably be spent on computer time for structural analysis cannot be readily answered. The total cost for structural analysis is usually determined on the basis of tradition at an early stage of planning. Generally, the amount spent on computer time is small in comparison with the total project cost, and more generous allotments may be available in the future. However, funds will remain limited and improvement of software efficiency will remain an important task.

Rayleigh-Ritz Analysis

The problem of excessive computer time for nonlinear static or dynamic analysis of complex structures has been recognized by several authors. As suggested in Refs. 4 and 5, this difficulty can be overcome through a return to the use of global shape functions in a Rayleigh-Ritz type of analysis. Considerable savings in computer time would be achieved because a superposition of global functions might, with sufficient accuracy, represent a complicated deformation pattern with very few degrees of freedom.

The Galerkin and Rayleigh-Ritz methods were the primary tools for buckling and nonlinear analysis before the high-speed computer was adapted for scientific computations. Eventually these methods were replaced by the finite-element method. One reason for this development is the ease by which finite-element computer codes can be applied to general-type structures. The difficulty in choosing the global functions for the Rayleigh-Ritz analysis presents a problem. In particular, it seems easier to judge whether a given finite-element discretization will yield satisfactory accuracy than it is to decide on the completeness of a set of global functions.

The problem of lack of generality of a Rayleigh-Ritz-based code can be overcome by combining it with the finite-element method. The freedoms of the system are the amplitudes of the global trial functions and a finite-element discretization is used for the numerical integration that determines the coefficients in the nonlinear algebraic equations. In a finite-element formulation of the problem, we have

$$M\ddot{X}+SX-F=0 \tag{1}$$

where M is the mass matrix, S a nonlinear algebraic stiffness operator, and F the vector of external forces. The vector X represents the freedoms in the finite-element formulation;

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that is, displacement or rotation components at the structural nodes.

A Rayleigh-Ritz-type formulation may be obtained by introduction into the finite-element formulation of the substitution

$$X = Tq \tag{2}$$

where each column in the matrix T represents one of the trial functions. A trial function is defined by the finite-element discretization, i.e., by the displacement components and rotations at structural nodes and by the local shape functions peculiar to the element. The components of the vector q are coefficients in a linear combination of trial functions representing solutions of the system. These coefficients are the degrees of freedom in a reduced nonlinear algebraic equation system obtained through substitution of Eq. (2) into Eq. (1) and summation over the elements. The ith equation is of the form

$$\bar{M}_i \ddot{q}_i + \Sigma A_{ij} q_j + \Sigma\Sigma B_{ijk} q_j q_k + \Sigma\Sigma\Sigma C_{ijk\ell} q_j q_k q_\ell = \bar{F}_i$$
$$(i,j,k,\ell = 1, I) \tag{3}$$

where $\bar{M}_i$ and $\bar{F}_i$ are generalized masses and forces corresponding to the ith trial function. The nonlinear terms derive from the stiffness operator.

Vibration or buckling modes computed from the finite-element formulation could be used as trial functions in the Rayleigh-Ritz analysis. Although this procedure appears to have been employed successfully (Ref. 5, for example), some uncertainty would be present in any new application. The high standard required of today's computerized analysis makes it mandatory that the accuracy of the solution can be controlled. Such control is easily introduced in a finite-element based Rayleigh-Ritz analysis.

At any load or time step, the vector q can be introduced in Eq. (2) to produce the current vector X of nodal displacements and rotations. Substitution of this X vector into the equilibrium equations for the discretized system [Eq. (1)] yields a vector R of residuals. This vector, representing the unbalance in each of the equations for the discrete system, gives directly a measure of the accuracy of the solution obtained from the reduced system. The solution of the reduced system represents an exact solution to the discrete system if $R = 0$ for all i.

Once the vector R is available, the nonlinear finite-element formulation can be used to compute the form of an additional trial function. Solution of the discrete system with the R vector as a right-hand side gives us a displacement vector that makes the set of trial functions complete, in the sense that it allows an exact solution of the finite-element formulation at the current time or load level. Consequently, we can periodically update our trial functions, so that any drift from the true solution is eliminated (minimized in the case of transient analysis).

Since the trial functions can be determined automatically for any discretized structure, both deficiencies of the Rayleigh-Ritz method can be overcome. A plausible logic for a computer code based on a Rayleigh-Ritz analysis with convergence control is illustrated by the flow diagram in Fig. 1. After control parameters, geometry, loads, and element configuration have been read in, an initial set of trial functions can be formed. The analyst has the option to define his own initial functions or to use the linear solution and a number of buckling or vibration modes obtained from the finite element formulation. Once the initial trial functions have been determined, generalized masses and forces, as well as the coefficients in the stiffness operator (A, B, and C in Eq. 3), are computed.

At every Mth (user input) load step, the residuals R_i can be computed in the discrete system. A norm N of the residuals may be defined, for example, as

$$N = \sum_{i=1}^{I} \left(\frac{R_i}{F_i}\right)^2 \tag{4}$$

where I is the number of freedoms in the discrete system. In the dynamic case, normalization may be used on inertia forces because all external forces may be zero (initial velocity). If the norm of the error exceeds an input value Q, the vector R is used as the right-hand side in the equation system for the finite-element system. The obtained solution is included, after orthonormalization, as an additional trial function. If the number of trial functions has reached a predetermined maximum, the new function can be added to one of the others before orthonormalization instead of being included as an independent trial function.

Circular Arch Analysis

The advantages of a Rayleigh-Ritz based finite-element program are certainly greatest for problems requiring a large number of freedoms in the discrete model. However, before

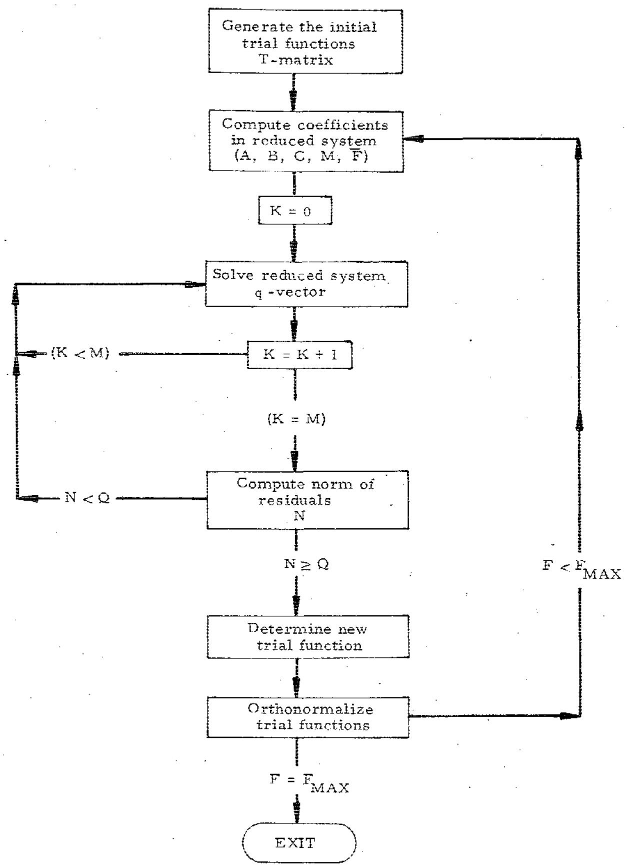


Fig. 1 Flow chart for Rayleigh-Ritz-based finite-element analysis.

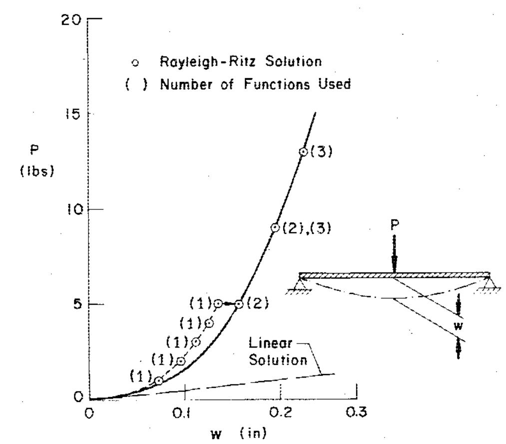


Fig. 2 Deformation of straight beam.

embarking on the development of general purpose programs with automatic generation of trial functions, it seems prudent to acquire some experience by application in a simple case. Therefore, a computer program has been derived that is restricted to analysis of arches and beams. Some of the results obtained are presented in the following paragraphs.

The first example consists of a straight beam with a point load midlength. The beam is free to rotate at the end points, but all end point displacements are restrained. The beam has a Young's modulus of 7,690,000 psi, a moment of inertia of 0.0001109 in.^{4}, and a length of 20 in. Results are shown in Fig. 2.

Here the linear solution alone constitutes the initial set of trial functions. The set is updated on every fifth time step, if the norm of the residuals exceeds the allowable magnitude. The load step is chosen so that the effect of nonlinearity is already considerable at the first load step. The linear analysis gives a shape function that minimizes the bending strain energy, while the energy due to stretching is ignored. The axial displacements are all zero in the trial function and, consequently, the stiffness of the beam is overestimated. Therefore, the linear analysis does not yield a particularly good trial function. At $P=5$ lb, the error in midpoint displacement is 15.9%. After a second function is added at $P=5$ lb, the Rayleigh-Ritz solution agrees exactly with the solution of the initial discrete system. As the load is increased, the error will again build up, but at the next return to the discrete system at $P=9$ lb, the error is only 0.2%. The convergence criterion was severe enough so that a third function was added. With three Rayleigh-Ritz functions, the displacement was found to be in error by only 0.004% at $P=13$ lb.

If more accurate solutions are desired at lower load levels, another function must be added earlier in the analysis.

In the second case, a circular arch was considered. Geometric and material data are given in Fig. 3. An inward-directed point force was applied at the midpoint of the arch. In a finite-element analysis with a true Newton method used for the solution of the nonlinear equations, convergence can be obtained up to a limit point at the load level 0.8155×10^6 lb. Efforts to solve the set of nonlinear equations for larger values of the load result in convergence failure. The true curve beyond the maximum can be computed by loading through controlled displacement.

The Rayleigh-Ritz analysis was first applied with only one trial function. Figure 3 shows the results obtained with the linear solution as a trial function (without updating). Also shown is a case in which only one trial function is included, but updating is allowed on every fifth step through addition of the displacement vector corresponding to the residuals in the discrete systems. The linear solution was again used as an initial trial function.

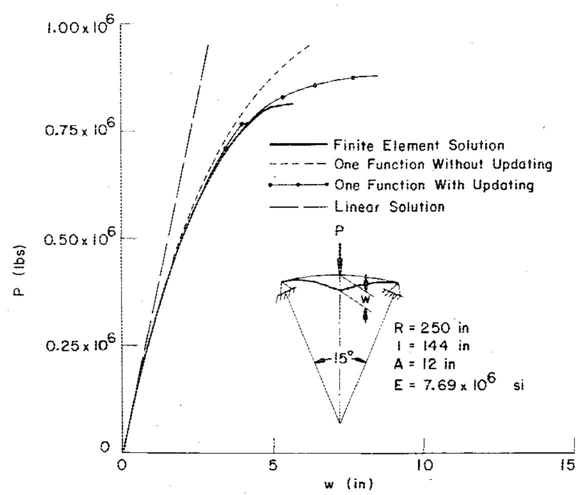


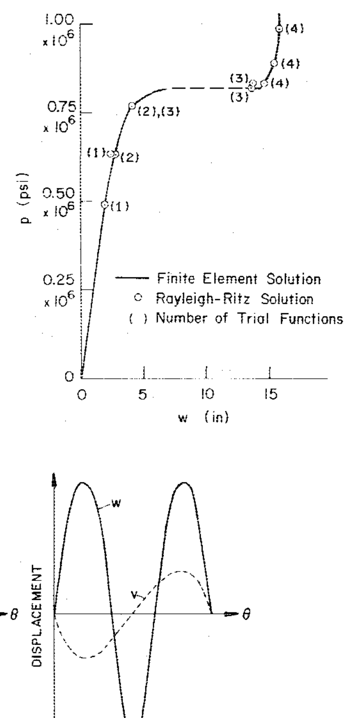
Fig. 3 Solutions with one trial function.

Figure 4 shows the results of an analysis in which the residuals are evaluated on every fifth step and a trial function is added, unless the convergence criterion is satisfied. The first results were obtained at $P=0.49 \times 10^6$ lb, with the linear solution as the only trial function. At this load level, the error is about 1.2%. The error grows gradually to 3% at the first load level at which the residual is checked, i.e., at $P=0.63 \times 10^6$ lb. At this point, a second trial function is added. From 0.63×10^6 to $P=0.77 \times 10^6$ lb, the error grows from 0 to 0.17%. This solution did not quite meet the convergence criterion and a third trial function was added. The solution obtained at $P=0.8247 \times 10^6$ lb is in the postbuckling range. The error at this point is 3.2% and a fourth trial function was added. The analysis was continued up to $P=0.98 \times 10^6$ lb. At this point, the error with four trial functions is still only 0.006%. The four trial functions are shown in Fig. 5.

The third example consisted of a circular arch clamped at both ends and subjected to a uniform inward pressure p. The arch has a radius of 250 in. and spans an angle of 15 deg. Its area is 0.42875 in.^{2}, the moment of inertia 0.00655 in.^{4}, and Young's modulus 10^7 psi. Numerical results for this example are available in Ref. 6.

If only one-half of the arch is considered and displacement symmetry is enforced at the midpoint, we can compute the load displacement relation corresponding to symmetric deformation. In Fig. 6, the midpoint displacement W_s for such deformation indicates a limit point for a pressure of about 5.56 psi. However, if the entire arch is included in the analysis so that antisymmetric deformation is allowed, we find that the determinant of the coefficient matrix changes

Fig. 4 Rayleigh-Ritz solution for circular arch.



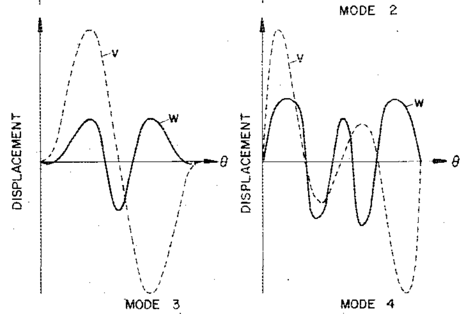


Fig. 5 The trial functions.

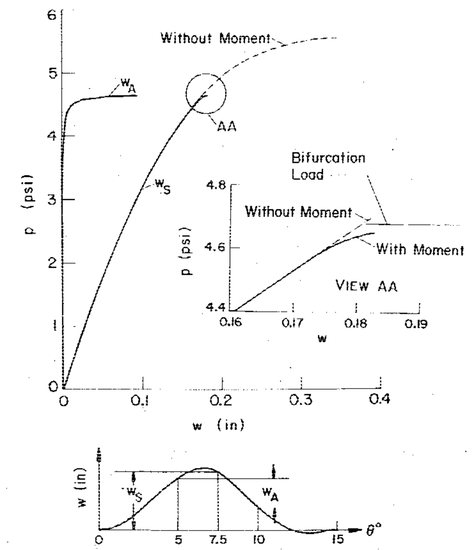


Fig. 6 Deformation of arch with uniform pressure.

sign at $p=4.68$ psi. This indicates that a bifurcation takes place at this load level. Computed values for the limit point as well as the bifurcation point agree well with the results of Ref. 6. If a small antisymmetric geometric imperfection is present, the antisymmetric deformation will grow rapidly as the bifurcation buckling load is approached.

The presence of a bifurcation into a new mode of deformation presents some difficulty in the use of the Rayleigh-Ritz analysis with automatic choice of trial functions. If no imperfections are included, the selected set of trail functions will only contain symmetric modes. It may be found on return to the discrete system that the norm of the residuals is acceptably small but the coefficient matrix has one or more negative roots. In that case, the solution of the reduced system corresponds to unstable equilibrium and it will be necessary to rerun the case either with some nonsymmetric mode included in the initial set or with a nonsymmetric imperfection included in the geometry.

In the Rayleigh-Ritz analysis of the arch, the effect of a geometric imperfection was introduced by the application of a small moment at midlength. In the first attempt, an initial loadstep of 0.25 psi was used and the accuracy check was performed at every fifth load step. The linear solution, slightly nonsymmetric because of the presence of the moment, was used as the only component of the initial set of trial functions. A second, basically symmetric vector was added at the first return to the discrete system ($p=2$ psi). An antisymmetric trial function was added at $p=3$ psi. At $p=4$ psi, a fourth function with a rather large antisymmetric part was added. Despite the presence of antisymmetry in the displacement pattern, convergence was obtained for loads up to $p=5$ psi. The accuracy check at this load level indicates that the coefficient matrix in the discrete system has a negative root. Therefore, it is necessary to return to $p=4$ psi and to use a smaller load step in the solution of the reduced system.

With a load step of 0.1 psi, it is found that the antisymmetric deformation grows rapidly before the bifurcation point is reached. The load step is adjusted automatically if nonconvergence occurs in the effort to solve the reduced system. After convergence was obtained at 4.65 psi but failed at 4.7 psi, the computations were discontinued. The lateral displacement along the arch is shown in Fig. 6. The amplitude of antisymmetric displacement W_A is represented here by the difference between the displacements at the symmetrically located points at $\theta=5$ and 10 deg. Symmetric and antisymmetric parts of the deformation according to the Rayleigh-Ritz analysis are also shown in Fig. 6. The amplitude of the antisymmetric deformation grows very rapidly at the bifurcation point and the amplitude of the symmetric part begins to deviate from the solution of the symmetric case.

Conclusions

A Rayleigh-Ritz-type analysis procedure intended for complex structures in the nonlinear domain has been proposed. The development of a computer program based on these ideas would be justified by substantial savings in computer time for large cases. In particular, the technique may make possible structural analyses that otherwise would not be economically feasible.

For demonstration, the analysis procedure was applied to the static analysis of beams and arches in the nonlinear range. Such simple structures are certainly not among those for which it is urgent to save computer time. However, they do exhibit a certain complexity in behavior, including stability failure both at limit points and bifurcation points. They are suitable, therefore, for use in a preliminary examination of the potential of the proposed method.

The cases for which the Rayleigh-Ritz analysis is contemplated are those in which the displacement pattern must be represented by upward of a thousand discrete degrees of freedom. With 4000 discrete freedoms and ten or less trial functions, for example, the computer time for computation of the coefficients in the reduced system is small in comparison to the solution time for the discrete system. The computer time for solution of the reduced system is negligible. Therefore, if new trial functions need to be added at each fifth step, we would have at least a fivefold saving in computer time without loss in accuracy. Especially for transient analysis, it appears that even more dramatic savings may be possible. The results of the analysis of beams and arches seem to be such that they encourage continued development. Even with 39 degrees of freedom in the initial system and up to six trial functions, the Rayleigh-Ritz-type analysis tended to require less computer time than the direct solution of the discrete system. However, emphasis was not placed on optimization of strategy and, therefore, we cannot make any reliable conclusions about the computer economy.

Application of the procedure in transient analysis may require some caution. In a nonlinear static analysis an accurate analysis can be obtained independently of the quality of the solution at previous load steps. In a transient analysis this is not the case—an error at an early stage leads to inaccurate initial conditions for the subsequent deformation history. Therefore, if at some time step the solution of the reduced system gives unacceptably large residuals in the discrete system, the analysis with an additional mode included should be restarted at the last value of time for which an acceptable solution was obtained.

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References

[1] Almroth, B. O. and Brogan, F. A., "Automated Choice of Procedures in Computerized Structural Analysis," *Computers and Structures*, Vol. 7, June 1977, pp. 335-342.

[2] Bushnell, D., "A Computerized Information Retrieval System," *Structural Mechanics Computer Programs*, edited by W. Pilkey, K. Saczalski, and H. Schaffer, University Press of Virginia, Charlottsville, Va., pp. 735-804.

[3] Almroth, B. O. and Brogan, F. A., "Bifurcation Buckling as an Approximation of the Collapse Load for General Shells," *AIAA Journal*, Vol. 10, April 1972, pp. 463-467.

[4] Almroth, B. O., "Structural Stability of Complex Structures," *ASCE Proceedings, 2nd Annual Engineering Mechanics Division Specialty Conference*, Raleigh, N.C., May 1977.

[5] Nagy, D. A., "Modal Representation of Geometrically Nonlinear Behavior by the Finite Element Method," Paper presented at the 4th International Conference on Structural Mechanics in Reactor Technology, San Francisco, Aug. 1977.

[6] Kerr, A. D. and Soifer, M. T., "The Linearization of the Prebuckling State and its Effect on the Instability Loads," *Journal of Applied Mechanics*, Vol. 36, Dec. 1969, pp. 775-783.